1 Introduction

You already know a lot of mathematics. You have been receiving formal training in mathematics since your first day of school and you have already mastered more mathematical skills than most of the humans who have ever lived. You already have extremely powerful tools. However, there are two questions that you have probably not answered (or maybe even been asked) in all your mathematics classes:

1. What is mathematics and how is it different from other subjects?

2. Why is mathematics important, or, how can I use the mathematics I already know to do interesting things?

The goal of this course is to begin to answer these questions.

For us “mathematics” means the “mathematical sciences”, which includes traditional mathematics along with statistics and computer science. These three areas share common roots and many common tools and techniques. Our examples will come from all three of these areas.

To paraphrase Shrek, mathematics is like an onion—mathematics does not smell bad when you leave it in the sun or make you cry—mathematics has layers. The process of doing mathematics is one of peeling back those layers, giving a more and more detailed understanding of some aspect of our world.

All good mathematics starts with a problem. The more interesting the problem, the more important the mathematics. Of course “interesting” is a relative term, what is interesting to me may not be interesting to you and vice versa. However, everybody has questions that are interesting to themselves and probably many others. The problem can be esoteric or practical, but the first step should always be choosing the problem.

The first step in attacking the problem is the key step and it is probably not something you have done very little in your study of mathematics. Almost every problem, and certainly every problem that talks about the “real world”, is incredibly complicated. For example, many on-line book and movie sellers have built in features that recommend titles that may interest you. If you think about this problem for a minute or two, it is easy to come up with hundreds of factors that might influence preferences. For example, the season (are you looking for a beach read or a serious text), the economy (you may want a book on luxury yachts or a history of the Great Depression), the time of day of your search (if you are searching at 12 PM on Saturday then you might prefer a book on social networking), and so on. Including all these factors is hopeless.

So the first step in attacking the problem must be to “make a model” or create an abstraction of your problem. By this we mean that you must decide what influences are the most important and precisely describe what situations you will consider. It is this model problem that you actually study. A good model walks the line between being simple and concrete enough to be understandable, but realistic enough to say interesting things about the original problem. Much of the “art” of quantitative reasoning goes into the construction of models and it is a process that is often repeated within the same problem. For example, to study how a car might skid on a slippery road, you might start by considering the motion of
a unicycle, then a bicycle and finally a car. We build up more accurate and more complicated models in layers.

Once the model has been precisely defined, then you can use the multitude of tools you already have to study the model. In mathematics this process starts by looking at examples, which is much like doing experiments in science. Next we make “conjectures”, statements that we think are true about the model and which we observe consistently in the examples. This is like creating “theories” in science. Next, and this is an aspect that is unique to mathematics, if we can, we “prove” the conjectures. Proofs are portrayed as the bogeyman of mathematics, but we will see that they are actually our saviors and ports in the storm.

Finally, we return to the original problem to see what we have learned. If the model is precise enough and our results complete enough, we may move on to the next problem. If not, then we add another layer, using what we have learned to make the model a more accurate characterization of the original problem.

Keep this process in mind as we progress through the course. The process of model building and creating layers of abstraction is tremendously powerful because it allows us to attack extremely difficult problems by a sequence of steps that are not so difficult.
2  A Toy Problem

Recall the outline of how mathematics is done from the previous section

\[
\begin{align*}
\text{Process} & \\
\text{Problem} & \quad \downarrow \\
\text{Model} & = \quad \text{Abstraction} \\
\downarrow & \\
\text{More Precise} & = \quad \text{Simpler} \\
\downarrow & \\
\text{Conjecture} & \rightarrow \quad \text{Proof} \\
\downarrow & \\
\text{Fame,} & \\
\text{Fortune,} & \\
\text{etc.} & \\
\downarrow & \\
\text{Repeat} & 
\end{align*}
\]

2.1  Stating the Problem

We start by considering a “toy problem”—that is, a problem that seems simple. This particular toy problem is about toys. Specifically, we look at different types of balls used in different sports. The surface of each of these balls is divided into pieces. We can think of each ball as being constructed of pieces. This is literally true soccer and footballs, but the pieces of the basketball are just drawn on the surface.

![Some toys](image)

Figure 1: Some toys

All the different balls are made up of different numbers of different shape pieces. The soccer ball is made up of pentagons and hexagons, the basketball is made of three sided pieces, deformed triangles, and the football is made up of pieces pictured in Fig. 2.

Our question is

*Can any collection of pieces be made into a ball?*
The immediate answer is “Yes”—we could take any collection of pieces and, overlapping them like paper mache, we can make a ball. (see Fig. 3). This is cheating. In the sports balls, the pieces are put together neatly along the edges. For example, if we start with a soccer ball and try to replace one of the pieces with a square piece, we can not match up the edges and corners of the square with the edges and corners of the hole (see Fig. 4).

![Figure 3: Pieces put together (following no rules) to make a ball.](image)

![Figure 4: Replacing a hexagon from a soccer ball with a square.](image)

### 2.2 Building the Model: Precision and Abstraction

The first order of business is to carefully state the rules for what constitutes a “piece” and the rules for gluing pieces together. It is important to remember that we make the rules. The rules help us understand exactly what we are doing, but they also limit the number of situations we are considering. So there is a trade off between generality and simplicity. To start, we want rules that allow all of the types of toy balls we have seen so far, but we don’t want to allow any sort of haphazard mushing together of pieces.

We start with the following:

**Rules for Pieces**

1. Pieces have no holes.
2. Pieces have sides (called “edges”). Each piece has at least two edges and at most a finite number of edges.

3. Each edge begins and ends at a special point called a vertex and the beginning and ending vertices are different.

4. Two edges of a piece touch only at a vertex.

Rules for Gluing:

1. Edges are glued together exactly two at a time with no overlap.

2. Vertices are glued only to vertices.

3. Balls are made with only finitely many pieces, so after gluing there are finitely many edges and finitely many vertices on the completed ball.

The gluing rules illustrate our point about making the problem abstract and precise at the cost of realism. For example, we can think of gluing pieces together along edges with no overlap, but in reality we would need at least a little overlap to apply the glue or the stitches.

The rules for pieces illustrate the balance between simplicity and generality. We could have restricted ourselves to pieces that were polygons, that is, pieces with edges that are line segments, but that eliminates pieces that make up the football. On the other hand, we can imagine pieces with many holes cut out. It turns out that pieces with holes are much more difficult to work with.

Being able to make up the rules gives us tremendous leeway in studying a problem. In some types of applications it seems like the rules have already been determined. For example, in studying the motion of the moon around the earth, Newton’s law of gravitational attraction has been known for hundreds of years. This is one rule we can not adjust to suit ourselves. However, we do get to decide if we want to include the effect of the sun on the earth-moon system. Including the sun turns the two body earth-moon problem into a much more difficult (but more accurate) three body problem. Even if we only consider the earth and the moon, we have the option of looking at the motion as if we are standing on the moon or the earth. The mathematics is very different depending on which point of view we choose.

2.3 Studying the Model

Now that we know the rules, we can look at examples to see if we can see any understandable patterns. We start with just counting the number of pieces, edges and vertices.

<table>
<thead>
<tr>
<th></th>
<th>Pieces</th>
<th>Edges</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Football</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Basketball</td>
<td>8</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Soccer ball</td>
<td>32</td>
<td>90</td>
<td>60</td>
</tr>
</tbody>
</table>

What patterns can we see in these numbers? Well, they are all even numbers, so we might suspect that the number of pieces, edges and vertices must always be even. That is,
we make a “Conjecture” that the numbers of pieces, edges and vertices must be even for every ball. Implicit in this conjecture is the word “always”. We are hoping that it is always true that if you make a ball out of pieces that follow the rules and using the rules of gluing, the resulting ball will always have an even number of pieces, edges and vertices.

This conjecture is false. While our examples above involve only even numbers, it is not too hard to think of an example like the triangular prism of Fig. 5. This is made of two triangles and three rectangles, so five pieces all together. When glued together there are nine edges and six vertices. This “counter example” shows that it is not always true that all three of the number of pieces, edges and vertices must be even on the finished ball.

Figure 5: Triangular prism made from pieces.

Adding a couple more examples, the triangular prism and the cube, to our chart, and after staring at it for a while, we start to see another interesting and more subtle pattern.

<table>
<thead>
<tr>
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<td>90</td>
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</tr>
<tr>
<td>Prism</td>
<td>5</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>Cube</td>
<td>6</td>
<td>12</td>
<td>8</td>
</tr>
</tbody>
</table>

(Don’t worry if this pattern hasn’t jumped out at you yet. People have been making balls for hundreds of years. If you looked at these numbers for hundreds of years, you would definitely discover the pattern without a hint.)

Notice that the number of edges is usually the largest number (but not always, for the football it is tied for largest). Even more, the number of pieces plus the number of vertices is almost equal to the number of edges. In fact, in every example above, the number of pieces plus the number of vertices is exactly two more than the number of edges. We can code up this observation into a simple formula

\[
\text{number of pieces} + \text{number of vertices} = \text{number of edges} + 2.
\]

If this were a science class, we would now have a “theory of gluing” that we would test with more experiments. We would have our students try all sorts of different ways to construct balls out of pieces following the rules and see if the formula always held. If it did, we would become more and more confident of our theory. We would announce our theory to the world. Then we would retire and wait for our Nobel prize.

We do the same in mathematics, look at more examples to see if our conjecture remains true. But just looking at examples is never completely satisfying. It may be that some clever
student will come up with some complicated construction where the pieces plus vertices equals edges plus four or minus three.

Even more troubling, while the formula holds in every case we have looked at, we do not know "why" it holds. We can't sleep at night. Our beautiful conjecture might abandon us at any moment, leaving us with nothing but a broken heart and the vague feeling of a wasted life.

In mathematics, we go one step beyond looking at examples. We give a proof.
3 Proof

If you have heard the word “proof” before, it was probably used by a nasty teacher or nastier babysitter to give you nightmares. For some reason, the idea of mathematical proof has gotten a very bad name.

This is too bad because a proof is nothing more or less than a convincing argument. This argument is based on the rules of logic and previous proven facts. The power of mathematical proof is not appreciated much by the general public, but it is one of the most important ways that mathematics differs from other subjects and it is one of the most powerful aspects of mathematics and, indeed, of human thought.

The mathematical facts proven by Isaac Newton over 300 years ago are the basics of Calculus and are taught to tens of thousands of students every year. Isaac Newton also did chemistry—or alchemy, trying to turn lead into gold. We do not remember this because, unlike his mathematical works, Newton did no proofs in alchemy.

In the days of ancient Greece, physics involved thinking of the four fundamental elements, Earth, Fire, Water and Air—and we now chuckle at their naivete. But the geometry of ancient Greece is taught every year to every high school student. This is because Euclid provided proofs—his theorems are as solid and true today as they were thousands of years ago.

So mathematical proofs are nothing to fear. In fact, a proof gives you security. It says to you “This statement is yours forever. It will stay with you through any trouble.” You should hold a proof close to your heart because it whispers to you in the cold lonely nights “I’ll never lie to you, I’ll always be here for you.” So we should do a proof.

3.1 Proof vs. Example

In science, there is no notion of proof. Theories stay theories, even when they are well-accepted by everyone (e.g., the Theory of Relativity or the Theory of Evolution). This is because there is no proof in science—only more and more evidence from more and more experiments. Eventually, the great preponderance of evidence is such that everyone accepts the Theory.

As we said above, “experiments” in mathematics are called “examples”, (although the term “numerical experiments” is a phrase that is becoming more common). However, because of the process of building precise models, we can go beyond the examples and experiments. To prove a statement, we give an argument based on the established rules for our model and on previously proven statements. A proof must apply to every situation covered by your statement. After a proof has been given, the results of subsequent experiments are determined. The experiments become applications because there is no doubt how they will come out—we have a proof!

Watch for the difference between proof and example in everyday conversation. For example

Person A: Does that store have good prices?
Person B: Yes, I got this Jacket for half of the price anyplace else.

This is an example. The question was about prices in general, but the answer only referred to one example. Even if the price on the jacket was great, it does not refer to other items.
On the other hand,

Person A: Does that store have good prices?
Person B: Yes, they guarantee to sell for half of the price anywhere else.

This is a proof, provided you accept the guarantee.

Part of the bad reputation that proofs get is because the first examples student are shown of proof seem to be proving the obvious (like H.G. Wells describing the work of Henry James—a hippopotamus laboring to pick up a pea). We give one example of a proof of an “obvious” fact:

Claim: You are mortal.

Proof: All people are mortal.

You are a person.

Therefore, you are mortal.

Here, the first two statements are really part of our model of humanity— that people are mortal and that you are a person are given or already proven. The conclusion, that you are mortal, follows directly from these statements. We can now elevate the statement that you are mortal from the level of claim to that of Theorem—which is a statement that has a proof. You may have already seen this proof with “you” replaced by “Socrates”—but since Socrates is already dead, there are other proofs.

Let’s return to our model for making balls out of pieces in order to give non-trivial examples of proofs.

3.2 Proving a Negative

We know that a proof will stay with us and not desert us, but, as in any relationship, to gain this level of trust we must be honest in return. When we say we are giving a proof, we are referring to a particular precise statement. We cannot prove a statement that is not clear or vague in any way, and we must be clear and honest about what we have proven.

Being precise does not, however, require that we be trivial. There are some amazing mathematical statements that have proofs. It is even possible to “prove a negative”, that is, prove that some action is impossible, even if we have not (or even could not) possibly try all different ways to do it.

As an example, we prove the following statement:

Claim: It is impossible to construct a ball, following the rules above, from five pentagons.

That is, no matter how you try gluing together the sides of five pentagons, following the rules, you will never be able to make a ball. This is quite a claim. There are many ways to try to glue the edges together, we are saying all at once that none of them will work. We could try a few possibilities and then say “see, it won’t work”, but that is just doing examples, not giving a proof. To give a proof that covers all possible ways to try to glue, we must be more clever.

Proof: First, let’s assume we could make a ball, following the rules, from five pentagons.

With five pentagons, we start with five times five or 25 edges. One of the rules is that edges are glued together exactly two at a time. But 25 is an odd number. If we glue edges
together two at a time, we are always end up with a leftover edge. It is impossible to glue together 25 edges two at a time and end up with no loose edges.

Hence, our original assumption, that it was possible to make a ball following the rules out of five pentagons, leads to the ridiculous statement that the completed ball has a loose edge that isn’t glued anywhere. But this is nonsense, balls do not have loose edges. This means that our original assumption that we can make a ball following the rules from five pentagons must be wrong. So, the opposite is true: It is impossible to make a ball following the rules from five pentagons.

In this proof we acted a bit like the hippopotamus with a pea—you could just as easily say: “Suppose you could make a ball from five pentagons. But then the 25 edges of the five pentagons would have to be glued together two at a time and this is impossible because 25 is odd. So it must be impossible to make a ball from five pentagons.”

Proofs do not have to be long, drawn out excruciatingly detailed arguments, repeating every step twice (that is just more bad press). Exactly how much detail you give depends on your audience. You must say things clearly enough so that your intended readers will understand.

The proof above is an example of “proof by contradiction”. The idea is to assume the opposite of what you want to prove, then show that this assumption leads to something ridiculous (like 25/2 is an integer). Once this contradiction is reached, you know that your original assumptions must have been incorrect and the opposite must be true.

Next we do a more complicated proof of a more amazing fact.

### 3.3 Proof by Induction

There are other types or techniques of proof, one of which we exploit to prove our conjecture concerning pieces, edges and vertices. This technique is called “induction”. It works by building up from simple to more complicated cases.

For example, if we start with a ball covered by two pieces, each piece having two edges and two vertices as shown below, then it is easy to count on the completed ball that pieces (2) + edges (2) = vertices (2) + 2.

![Simple case with two pieces, two edges and two vertices.](image)

Figure 6: Simple case with two pieces, two edges and two vertices.

Now we consider how we can use the formula in this simple case to help show it in more complicated cases. We do this by considering how we can make the simple case above more complicated.

We list the ways we can make a simple construction of a ball more complicated, always sticking with the rules of pieces and the rules of gluing.
1. We can add a vertex in the middle of an existing edge.

This increases the number of vertices by one since we are adding a new vertex, but it also increases the number of edges by one since we split one edge into two. So if the formula

\[ \text{pieces} + \text{vertices} = \text{edges} + 2 \]

is true to start and we add one to the number of vertices and one to the number of edges, the formula is still true.

2. We can add an edge connecting two existing vertices.

This leaves the number of vertices unchanged, but it adds one edge and it cuts one of the pieces into two, increasing the number of pieces by one. So again, if

\[ \text{pieces} + \text{vertices} = \text{edges} + 2 \]

is true to start then it is still true after the edge is added (we add one to the number of pieces and one to the number of edges).

3. We can add an edge that starts at an existing vertex and ends on an edge, creating a new vertex.
Here we have increased the number of pieces by one (cutting one piece into two), we have added a vertex and we have increased the number of edges by two (the new edge and the new vertex cuts an existing edge into two). So again, if

$$\text{pieces} + \text{vertices} = \text{edges} + 2$$

is true to start then it is still true after the edge is added, (we add one to the number of pieces, one to the number of vertices and two to the number of edges).

4. Finally, we can add an edge that starts at a new vertex along an existing edge and ends at a new vertex along an existing edge.

![Figure 10: Adding an edge and two new vertices.](image)

In this case, we have one more piece, three more edges and two more vertices. So yet again, if

$$\text{pieces} + \text{vertices} = \text{edges} + 2$$

is true to start then it is still true after the edge is added.

Since the formula holds for the simplest case and the formula continues to be true whenever we add a vertex or edge, we see that the formula always holds.

The first step of the proof, verifying the formula for the simplest case, is called the “basis” step. Showing the the formula remains true when adding a vertex or an edge is called the “induction” step. These two steps together give what we call an “induction proof”.

As we said above, if we expect a theorem to always be honest with us, then we must be completely honest in return. What we have proven is that the formula holds for any construction of a ball that can be created using the steps above. Can every decomposition of a ball into pieces following the original rules be constructed using these steps? Claiming this also requires a proof. If you would like to think about how to prove this, think about starting with the completed ball and removing edges and vertices one at a time so that the piece and gluing rules remain true.

### 3.3.1 Celebration

By providing a proof, we have turned our formula from conjecture into something more—a “theorem”. We can now use this formula without fear. Moreover, we have some understanding of why the formula holds.

Since we now have a proof, we can now give the formula a name—"Euler’s Formula". Euler was one of those dead, European mathematicians of the 1700’s that did so much great math.
3.4 More Theorems

Now that we know that
\[ \text{pieces} + \text{vertices} = \text{edges} + 2 \]
we can use this to prove more theorems.

For example:

Claim: For a ball made following the rules, at least one of the number of pieces, edges or vertices on the completed ball must be an even number.

Proof: Since we know that
\[ \text{pieces} + \text{vertices} = \text{edges} + 2 \]
if both the number of pieces and the number of vertices on the completed ball are odd, then we know that
\[ \text{pieces} + \text{vertices} \]
must be even, since the sum of two odd numbers is even. But then
\[ \text{pieces} + \text{vertices} - 2 \]
must also be even, because an even number minus two is still even. But we know from Euler’s formula that
\[ \text{pieces} + \text{vertices} - 2 = \text{edges} \]
so the number of edges must be even.

Hence, we can not have all three of pieces, edges and vertices odd at the same time.

Note that other proofs were possible. We could have started out saying suppose the number of vertices and edges is odd, then show that the number of pieces must be even. We could even have given a proof by contradiction by assuming all three were odd then get a contradiction from the fact that Euler’s formula would give an even number equal to an odd number.

From this we can immediately say that it is impossible to make a ball following the rules such that the completed ball has 15 pieces, 23 edges and 11 vertices. We do not have to think about what shape the pieces could be or how they could be glued, because all three numbers are odd, we know this is impossible—end of story.

4 Application: Platonic Solids

In discussion section you looked at an extension of the gluing rules as follows:

1. All the previous rules hold
2. All the pieces are the same shape polygon.
3. Every vertex on the completed ball touches the same number of pieces.
4. The completed ball has more than two pieces.
The balls made following these rules are called “Platonic Solids”. You saw in discussion that these are not so easy to build. You can not just take any bunch of polygons and make a Platonic solid out of all of them. It turns out that Euler’s formula is just the tool for understanding Platonic solids. We will also use a little bit of the algebra you learned in high school—this should make you happy! You finally get to use it for something!

Suppose we want to build a Platonic solid out of squares. We know we can make a cube, which is made of six squares and has exactly three squares touching each vertex on the completed solid. Is this the only way to make a Platonic solid out of squares, or is there some other way involving more total squares and more squares touching each vertex? How can we tell without trying all possibilities?

Luckily, Euler’s formula will help us out. If you are going to build a Platonic solid out of squares, then you will start with some number, say \( S \) of squares. Before gluing you will have \( 4S \) edges and \( 4S \) vertices. After gluing, there will be \( 4S/2 = 2S \) edges, because edges are glued two at a time.

The situation for vertices is not so clear. By special rule number three above, we know that every vertex on the completed Platonic solid touches the same number of squares. Let’s give this number a name, say \( Q \). Now we know that the number of vertices on the completed ball will be \( 4S/Q \), since the available vertices are glued together in groups of size \( Q \).

Now we turn to Euler’s Formula. We know that

\[
\text{pieces} + \text{vertices} = \text{edges} + 2
\]

and in this case that means

\[
S + \frac{4S}{Q} = 2S + 2.
\]

Now for the algebra, this gives

\[
S + \frac{4S}{Q} - 2S = 2,
\]

or

\[
\frac{4S}{Q} - S = 2,
\]

or

\[
S \left( \frac{4}{Q} - 1 \right) = 2.
\]

This says something really important. Euler’s formula tells us that we can not pick any numbers for \( S \), the number of squares, and \( Q \), the number of squares touching each vertex. These two numbers have to be related to each other, specifically by the formula above.

So what choices are possible for \( S \) and \( Q \)? Well, \( Q = 1 \) does not make sense—if a vertex were glued only to itself, it would not be glued to anything and our finished ball would have a loose vertex. So \( Q = 1 \) is rejected. This would violate rule 4 requiring more than two pieces. If \( Q = 2 \) then we get

\[
S \left( \frac{4}{2} - 1 \right) = 2
\]

so

\[
S(2 - 1) = 2,
\]
or

\[ S = 2 \]

and we are violating rule 4 that we must have more than two pieces.

We know it is possible to have \( Q = 3 \) since this is what occurs in the cube. Taking \( Q = 3 \), we get

\[ S \left( \frac{4}{3} - 1 \right) = 2, \]

or

\[ S \frac{1}{3} = 2 \]

or

\[ S = 6, \]

exactly as in the cube.

For \( Q = 4 \), we get

\[ S \left( \frac{4}{4} - 1 \right) = 2 \]

or

\[ S \cdot 0 = 2 \]

and this is just impossible. If \( Q = 4 \) there is no way to choose \( S \) so that Euler’s formula holds!. Hence \( Q = 4 \) must be impossible.

Similarly, if \( Q > 4 \) then

\[ \frac{4}{Q} < 1, \]

so

\[ \frac{4}{Q} - 1 < 0 \]

and there is no way we can have

\[ S \left( \frac{4}{Q} - 1 \right) = 2 \]

since \( S \) must be a positive integer! So again, we see that \( Q > 4 \) must be impossible. Hence, we know that the only way to make a Platonic solid out of squares is to use six squares and to have three squares touching each vertex.

We are not quite all the way to saying that the only Platonic solid makable from squares is a cube (honest relationships!). What we know is that exactly six squares must be used and every vertex attaches to three squares, so the possibilities have gone from infinitely many to a number we can handle.

Let’s try the same idea for triangles.

To make a Platonic solid out of triangles, you will have some number of triangles, say \( T \) to start with. These triangles have a total of \( 3T \) edges and \( 3T \) vertices before gluing. We know that the edges are glued together two at a time, so the completed ball will have \( 3T/2 \) edges.

We also know, from the special rule number 3 above for Platonic solids, that every vertex on the completed ball will touch the same number of triangles. Let’s give that number a
name, say $Q$. So the completed Platonic solid will have $3T/Q$ vertices since the vertices are glued together in groups of $Q$ at each vertex.

Again, Euler’s formula, 
\[
\text{pieces} + \text{vertices} = \text{edges} + 2
\]
becomes
\[
T + \frac{3T}{Q} = \frac{3T}{2} + 2.
\]
This means that
\[
T + \frac{3T}{Q} - \frac{3T}{2} = 2,
\]
so
\[
T \left(1 + \frac{3}{Q} - \frac{3}{2}\right) = 2,
\]
or
\[
T \left(\frac{3}{Q} + 1 - \frac{3}{2}\right) = 2,
\]
which gives
\[
T \left(\frac{3}{Q} - \frac{1}{2}\right) = 2.
\]

Again, we see that not every combination of $T$ and $Q$ is possible. In the exercises you will show that, in fact, there are only three combinations of $T$ and $Q$ that can be used to make a Platonic solid out of triangles.

## 5 Conclusion

We have come full circle. We started talking about making toy balls out of pieces of material or plastic. We made a model with precise, but reasonable rules for how such balls are put together. Based on examples, we were able to see subtle patterns in the constructions. We proved that some of these patterns always hold—so we know that these apply to every ball. Finally, we used that knowledge to show the non-trivial fact that there are a very limited number of ways to build Platonic Solids. We could continue indefinitely with this topic as it is the gateway to a great deal of mathematics.

However, for this class, there are two important take away lessons from this first section. First is the process. We will repeat this process for a number of different examples throughout the course.

The second lesson is the power of mathematical proof. A proof is not a magic word that gives certain wizards special powers. A proof is not just one more example. A proof is a convincing argument. Like any story, a proof has a beginning, what you already know, a middle, where you make deductions from what you know, and an end, where you celebrate an extension of your knowledge. We will see a number of proofs through the semester—watch for them and value them as you would an honest friend.